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On the Efficiency of General Rational Approximation

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Let $0 \leqslant \lambda_1 < \lambda_2 < \cdots < \lambda_n$ be a finite set of real numbers and let $\mathscr{R}_\Lambda$ denote the set of rational functions of the form

$$R(x) = \frac{P(x)}{Q(x)} = \frac{\sum a_k x^{\lambda_k}}{\sum b_k x^{\lambda_k}}.$$

We seek to determine the degree of approximation possible by functions in $\mathscr{R}_\Lambda$ to arbitrary continuous functions on [0, 1]. More precisely, we seek upper bounds for the approximation index I_Λ defined as

DEFINITION.

$$I_\Lambda = \sup_{f \in \mathscr{S}} \inf_{R \in \mathscr{R}_\Lambda} \| f - R \|$$

where $\mathscr{S}$ denotes the set of contractions on [0, 1], that is, the set of functions f satisfying $|f(x) - f(y)| \leqslant |x - y|$ for all $0 \leqslant x < y \leqslant 1$, and $\| \cdot \|$ denotes the usual sup norm.

The importance of I_Λ in approximating an arbitrary continuous function lies in the fact that for any continuous f, there is some $R \in \mathscr{R}_\Lambda$ such that

$$\| f - R \| \leqslant 2\omega_f(I_\Lambda)$$

where ω_f denotes the modulus of continuity of f. (See, e.g. [1, p. 440].)

It can be shown by a standard argument that $I_\Lambda \geqslant 1/2n$. (Consider the $h \in \mathscr{S}$ satisfying $h(x) = (-1)^k/2n$ for $x = k/n$, $k = 0, 1,..., n$, and linear in between; then apply Descartes' Rule of Signs to show that $R(x) \equiv 0$ gives the best uniform approximation in $\mathscr{R}_\Lambda$.) On the other hand, it was recently shown [2] that for any infinite sequence $\lambda_1, \lambda_2, \lambda_3,...,$ with $\lambda_k \to \infty$ the set of rational combinations of the monomials x^λ is dense in $C[0, 1]$. It follows that for any such sequence, $I_\Lambda \to 0$ as $n \to \infty$. In fact, using a variation of the method in [2], we will be able to show that if the λ's are sufficiently separated, $I_\Lambda \leqslant 10/n$. Hence, in those cases, the order of magnitude of I_Λ is completely determined. Our main results are as follows.

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THEOREM. *Set* $\alpha_1 = \lambda_1$; $\alpha_k = \lambda_k - \lambda_{k-1}$ $k = 2, 3, \ldots, n$.

(A) *If* $\alpha_k \geqslant k$ *for all* $k \geqslant 2$

$$I_A \leqslant 10/n.$$

(B) *If, for* $k \geqslant 2$

(i) $\alpha_k \geqslant 1$

or

(ii) $\{\alpha_k\}$ *is monotonic*

$$I_A \leqslant A/n^{1/2}$$

where A *is a positive number which depends only on* α_2. (*In fact, it can be shown that* $A \leqslant 70$ Max $(1, 1/(\alpha_2)^{1/2})$.)

Note that (B) includes the cases where $\{\lambda_k\}$ is any subset of the integers or any "familiar" sequence such as $\lambda_k = k^p$, $\lambda_k = k \log k$, etc.

In our proof, we will construct rational functions of the form P/Q where $Q(x) > 0$ for $x > 0$. However, if $\lambda_1 > 0$, it will follow of course that $Q(0) = 0$ and by $R(0)$ we will understand $\lim_{x\to 0} R(x)$. (Alternatively, we could insist that $\lambda_1 = 0$.)

Proof of (A). To simplify notation, for any $f \in \mathscr{S}$, we consider $g(x) = nf(x/n)$ on the interval $[0, n]$. Note that

$$|g(x+\delta) - g(x)| \leqslant \delta. \tag{1}$$

We seek a rational function r such that $|g(x) - r(x)| \leqslant 10$ for all $x \in [0, n]$. Setting $R(x) = (1/n)\, r(nx)$, it will follow that $\|f - R\| \leqslant 10/n$. To construct r, let $g_j = g(j)$, $j = 1, 2, \ldots, n$, and set

$$r(x) = \sum_{j=1}^{n} \frac{g_j x^{\lambda_j}}{1^{\alpha_1} 2^{\alpha_2} \cdots j^{\alpha_j}} \Big/ \sum_{j=1}^{n} \frac{x^{\lambda_j}}{1^{\alpha_1} 2^{\alpha_2} \cdots j^{\alpha_j}}$$

so that

$$g(x) - r(x) = \sum (g(x) - g_j) \frac{x^{\lambda_j}}{1^{\alpha_1} 2^{\alpha_2} \cdots j^{\alpha_j}} \Big/ \sum \frac{x^{\lambda_j}}{1^{\alpha_1} 2^{\alpha_2} \cdots j^{\alpha_j}}. \tag{2}$$

Suppose then that $k - 1 \leqslant x \leqslant k$. To estimate $|g(x) - r(x)|$, we apply a triangle inequality in (2) and inequality (1) to obtain

$$|g(x) - r(x)|$$

$$\leqslant \sum_{j=1}^{k-1} |j - k| \frac{x^{\lambda_j}}{1^{\alpha_1} 2^{\alpha_2} \cdots j^{\alpha_j}} + \sum_{j=k}^{n} \frac{|j - k + 1|\, x^{\lambda_j}}{1^{\alpha_1} 2^{\alpha_2} \cdots j^{\alpha_j}} \Big/ \sum_{j=1}^{n} \frac{x^{\lambda_j}}{1^{\alpha_1} 2^{\alpha_2} \cdots j^{\alpha_j}}$$

$$\leq \sum_{j=1}^{k-1} \frac{|j-k|\,x^{\lambda_j}}{1^{\alpha_1}2^{\alpha_2}\cdots j^{\alpha_j}} \Big/ \frac{x^{\lambda_{k-1}}}{1^{\alpha_1}\cdots (k-1)^{\alpha_{k-1}}}$$

$$+ \sum_{j=k}^{n} \frac{|j-k+1|\,x^{\lambda_j}}{1^{\alpha_1}2^{\alpha_2}\cdots j^{\alpha_j}} \Big/ \frac{x^{\lambda_k}}{1^{\alpha_1}\cdots k^{\alpha_k}}.$$

If we call the first sum above S_1 and estimate the terms from $j = k-1$ to $j = 1$, we find

$$S_1 \leq 1 + 2\left(\frac{k-1}{x}\right)^{\alpha_{k-1}} + 3\left(\frac{k-2}{x}\right)^{\alpha_{k-2}}\left(\frac{k-1}{x}\right)^{\alpha_{k-1}} + \cdots$$

$$+ (k-1)\left(\frac{2}{x}\right)^{\alpha_2}\cdots\left(\frac{k-1}{x}\right)^{\alpha_{k-1}}$$

$$\leq 1 + 2 + 3\left(\frac{k-2}{k-1}\right)^{k-2} + \cdots + (k-1)\left(\frac{2}{k-1}\right)^{2}\cdots\left(\frac{k-2}{k-1}\right)^{k-2}.$$

Note then that $((k-j)/(k-1))^{k-j} \leq \frac{1}{2}$ for all $1 < j < k$ so that

$$S_1 \leq 1 + 2 + 3\cdot\frac{1}{2} + 4\cdot\frac{1}{2^2} + \cdots$$

$$= 1 + \sum_{i=0}^{\infty} \frac{(i+2)}{2^i}$$

$$= 7.$$

Similarly, setting

$$S_2 = \sum_{j=k}^{n} \frac{|j-k+1|\,x^{\lambda_j}}{1^{\alpha_1}\cdots j^{\alpha_j}} \Big/ \frac{x^{\lambda_k}}{1^{\alpha_1}\cdots k^{\alpha_k}}$$

we find

$$S_2 \leq 1 + 2\left(\frac{x}{k+1}\right)^{\alpha_{k+1}} + 3\left(\frac{x}{k+1}\right)^{\alpha_{k+1}}\left(\frac{x}{k+2}\right)^{\alpha_{k+2}} + \cdots$$

$$\leq 1 + 2\left(\frac{k}{k+1}\right)^{k+1} + 3\left(\frac{k}{k+1}\right)^{k+1}\left(\frac{k}{k+2}\right)^{k+2} + \cdots$$

and using the fact that $(k/(k+j))^{k+j}$ is bounded by $1/e$,

$$S_2 \leq \sum_{i=0}^{\infty} \frac{(i+1)}{e^i} = \frac{e^2}{(e-1)^2} < 3.$$

Hence

$$|g(x) - r(x)| \leq S_1 + S_2 < 10$$

anh (A) is proven.

Proof of (*B*). (i) If $\alpha_k \geqslant 1$, we need only consider the subsequence $\{\eta_k\}$ of $\{\lambda_k\}$, defined by

$$\eta_k = \lambda_{k(k+1)/2}, \qquad k = 1, 2, \ldots, [n^{1/2}].$$

The η_k's are sufficiently separated so that we can apply our results in (A) to conclude that for any $f \in \mathscr{S}$, there is some

$$R(x) = \sum a_k x^{\eta_k} \Big/ \sum b_k x^{\eta_k}$$

with

$$\| f - R \| \leqslant 10/[n^{1/2}] < 20/n^{1/2}.$$

Since the η's form a subset of the λ's, we deduce the same upper bound for I_Λ.

(ii) Suppose $\{\alpha_k\}_{k=2}^n$ is monotonic and suppose first that it is an increasing sequence. Then, if $\alpha_{[n/2]} \geqslant 1$, we can apply the previous results to $\lambda_{[n/2]}, \lambda_{[n/2]+1}, \ldots, \lambda_n$ to conclude $I_\Lambda \leqslant 30/n^{1/2}$. If, on the other hand, $\alpha_{[n/2]} < 1$ we consider the sequence $\eta_k = \lambda_k - \lambda_1$, $k = 1, 2, \ldots, [n/2]$. Since $\eta_1 = 0$ and $\eta_{k-1} - \eta_k \leqslant 1$, the analogous approximation index for "polynomials" $\sum a_k x^{\eta_k}$ is asymptotic to $(\sum \eta_k)^{-1/2} \leqslant ([n/2]\,\alpha_2)^{-1/2} \leqslant A/n^{1/2}$. See [3, p. 340]. That is, for any $f \in S$, we can find $P(x) = \sum a_k x^{\eta_k}$ such that

$$\| f - P \| \leqslant A/n^{1/2}.$$

Noting, then, that

$$P(x) = \sum a_k x^{\eta_k} = \sum a_k x^{\lambda_k} \Big/ x^{\lambda_1} \in \mathscr{R}_\Lambda,$$

our result follows. Completely analogous reasoning applies if $\{\alpha_k\}_2^n$ is decreasing and the proof is complete.

Remarks. (1) As in the proof of Theorem (B), it is evident that rational combinations of $\{x^{\lambda_k}\}$ form a dense set in $C[0, 1]$ for *any* sequence $0 \leqslant \lambda_1 < \lambda_2 < \cdots$ (even without assuming that $\lambda_k \to \infty$ as $k \to \infty$).

(2) While the upper bound in Theorem (B) depends on α_2 (or λ_2) this dependence may be unavoidable. In fact, if we take a decreasing sequence $\{\lambda_k\} \to 0$ (e.g., $\lambda_k = 1/2^k$) so that the λ_k's are not bounded away from zero, it is not even clear that rational combinations of the $\{x^{\lambda_k}\}$ are dense in $C[0, 1]$.

(3) As mentioned above, the exact order of magnitude of I_Λ in general is still undetermined. An appealing conjecture is that $I_\Lambda \sim 1/n$ for *any* sequence $0 \leqslant \lambda_1 < \lambda_2 < \cdots < \lambda_n$. Aside from the cases considered in (A), this is certainly true if $\lambda_k = \beta k$, $0 < \beta \leqslant 2$ since the corresponding polynomial approximation index is $\sim(1/n)$. If $\lambda_k = \beta k$, $\beta > 2$, the polynomial

approximation index is $\sim(1/n^{2/\beta})$. In these cases, the following smaller estimate can be given for I_A.

PROPOSITION. *If* $\lambda_k = \beta k$, $\beta > 2$,

$$I_A \leq \frac{A \log^2 n}{n}$$

where A depends only on β.

Proof. For any $f \in \mathscr{S}$, we consider

$$g(x) = f(x^{1/\beta}).$$

Since

$$|g(x+\delta) - g(x)| \leq |(x+\delta)^{1/\beta} - x^{1/\beta}| \leq \delta^{1/\beta},$$

g satisfies a Lipschitz condition of order $1/\beta$. Furthermore, g is of bounded variation since for any partition $0 = x_0 < x_1 < \cdots < x_m = 1$ of the unit interval

$$\sum_{k=1}^{m} |g(x_k) - g(x_{k-1})| \leq \sum (x_k^{1/\beta} - x_{k-1}^{1/\beta}) = 1.$$

But then, according to a theorem of Freud [4] there exists an ordinary nth degree rational function

$$R_n(x) = \frac{\sum_{k=1}^{n} a_k x^k}{\sum_{k=1}^{n} b_k x^k}$$

with

$$\|g - R\| \leq \frac{A \log^2 n}{n}.$$

We need only note then that

$$\|g - R\| = \left\| f(x^{1/\beta}) - \frac{\sum a_k x^k}{\sum b_k x^k} \right\| = \left\| f(x) - \frac{\sum a_k x^{\beta k}}{\sum b_k x^{\beta k}} \right\|$$

and the proposition is proven.

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